ON A TRANSFORMATION OF THE SECULAR EQUATION

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Starting from considerations used in the theory of integration of systems of linear differential equations, Krylov [1] has developed a method of reducing the determinant $|A - \lambda E|$ to a form, where λ occurs in the elements of one row only. Krylov's transformation has been analysed algebraically in a number of publications [2], [3], [4]. The present note offers a new and entirely elementary method of such a transformation. It does not require any knowledge of auxiliary material and we believe that it is the most purely algebraical of all known methods.

1. Let A be a real matrix of the n-th order, while x is a real column vector (with n components). Assume that the column vectors

$$x, Ax, A^2x, \ldots, A^{n-1}x \tag{1}$$

are linearly independent. Then the matrix

$$X = \| x, Ax, A^2x, \ldots, A^{n-1}x \| \tag{2}$$

is nonsingular. We form the product of the matrices

$$(A - \lambda E) X = \| Ax - \lambda x, \quad A^2x - \lambda Ax, \ldots, A^n x - \lambda A^{n-1}x \| \tag{3}$$

Turning to the determinants, we obtain by means of "fringeing"

$$|A - \lambda E| \; |X| = \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 \\ x & Ax - \lambda x & A^2x - \lambda Ax & \ldots & A^n x - \lambda A^{n-1} x \end{vmatrix} \tag{4}$$

Now we transform the obtained determinant in the following manner. We multiply the first column by λ and we add the result to the second column; we multiply the obtained second column by λ and we add the result to the third column; we multiply the obtained third column by λ and we add the result to the fourth column, and so forth In this way we ultimately arrive at

$$|A - \lambda E| \; |X| = \begin{vmatrix} 1 & \lambda & \lambda^2 & \ldots & \lambda^n \\ x & Ax & A^2x & \ldots & A^n x \end{vmatrix} \tag{5}$$

Since $|X| \neq 0$, we have

$$| A - \lambda E | = \frac{1}{|X|} \begin{vmatrix} 1 & \lambda & \lambda^2 & \ldots \lambda^n \\ x & Ax & A^2x \ldots A^n x \end{vmatrix} \tag{6}$$

In the determinant thus obtained we find λ in the elements of the first line only. This is Krylov's transformation in the so-called "regular" case [5].

2. The vector $A^n x$ can be represented as a linear combination of the vectors (1):

$$A^n x = p_1 A^{n-1} x + p_2 A^{n-2} x + \ldots + p_{n-1} Ax + p_n x \tag{7}$$

We introduce the notation

$$\lambda^n - p_1 \lambda^{n-1} - p_2 \lambda^{n-2} - \ldots - p_{n-1} \lambda - p_n = f(\lambda)$$

Multiplying in the determinant (6) the first, second, third, ..., n-th columns by $-p_n$, $-p_{n-1}$, $-p_{n-2}$, ..., $-p_1$, respectively, and adding the results to the last column, we find

$$| A - \lambda E | = \frac{1}{|X|} \begin{vmatrix} 1 & \lambda & \lambda^2 & \ldots & \lambda^{n-1} & f(\lambda) \\ x & Ax & A^2x & \ldots & A^{n-1} x & 0 \end{vmatrix} \tag{8}$$

Hence

$$| A - \lambda E | = (-1)^n f(\lambda) \tag{9}$$

Thus, for an expansion of the determinant $|A - \lambda E|$ in terms of the elements of the first line, it is sufficient to find the coefficients of the relation (7). It is known [5] that this can be done without computing the determinants.

3. Let us consider the so-called "singular" case, when the vectors (1) are linearly dependent at any choice of x. Assume that the vectors

$$x,\ Ax,\ A^2x, \ldots, A^{s-1}x \qquad (s < n) \tag{10}$$

are linearly independent, but

$$A^s x = q_1 A^{s-1} x + q_2 A^{s-2} x + \ldots + q_{s-1} Ax + q_s x \tag{11}$$

We will show that the polynomial

$$\psi(\lambda) = \lambda^s - q_1 \lambda^{s-1} - q_2 \lambda^{s-2} - \cdots - q_{s-1} \lambda - q_s \tag{12}$$

is a divisor of $|A - \lambda E|$.

Assume that in the matrix

$$\| x,\ Ax,\ A^2x, \ldots A^{s-1}x \| \tag{13}$$

of rank s, the rows with the subscripts m_1, m_2, ..., m_s are linearly independent. Denote by v_1, v_2, ..., v_{n-s} the subscripts of the remaining rows and by E_k the k-th column of the unit matrix E.

Form the nonsingular matrix

$$X = \| x, Ax, A^2x, \ldots, A^{s-1}x, E_{v_1}, E_{v_2}, \ldots, E_{v_{n-s}} \| \tag{14}$$

and consider the product

$$(A - \lambda E) X = \| Ax - \lambda x, A^2x - \lambda Ax, \ldots, A^s x - \lambda A^{s-1}x, z_{v_1}, z_{v_2}, \ldots, z_{v_{n-s}} \| \tag{15}$$

Turning to the determinants and "fringeing", we obtain, with the already known elementary transformations of the columns,

$$|A - \lambda E| \; |X| = \begin{vmatrix} 1 & \lambda & \lambda^2 \ldots \lambda^{s-1} & \lambda^s & 0 & 0 \ldots 0 \\ x & Ax & A^2x \ldots A^{s-1}x & A^s x & z_{v_1} & z_{v_2} \ldots z_{v_{n-s}} \end{vmatrix} \tag{16}$$

By means of elementary transformations, applied to the last n rows, this determinant can be given the form

$$\begin{vmatrix} 1 \;\; \lambda \;\; \lambda^2 \ldots \lambda^s & 0 \;\; 0 \ldots 0 \\ X_{s,\, s+1} & X_{s,\, n-s} \\ 0_{n-s,\, s+1} & X_{n-s,\, n-s} \end{vmatrix} \tag{17}$$

where all elements of the block $0_{n-s,\, s+1}$ are zero.

Since $|X| \neq 0$, we derive from (16)

$$|A - \lambda E| = \frac{|X_{n-s,\, n-s}|}{|X|} \Delta, \qquad \Delta = \begin{vmatrix} 1 \;\; \lambda \;\; \lambda^2 \ldots \lambda^s \\ \cdots\cdots\cdots \\ X_{s,\, s+1} \end{vmatrix} \tag{18}$$

From the first of these two equalities we see that the determinant Δ is a divisor of $|A - \lambda E|$.

We now note that the columns of the block $X_{s,\, s+1}$ obey the same linear relations, which are fulfilled by the columns of the matrix (13); therefore the determinant Δ can be reduced, by means of elementary transformations of the columns, to the form

$$\begin{vmatrix} 1 \;\; \lambda \;\; \lambda^2 \ldots \lambda^{s-1} & \psi(\lambda) \\ X_{s,s} & 0 \end{vmatrix} \tag{19}$$

Thus

$$|A - \lambda E| = (-1)^s \frac{|X_{n-s,\, n-s}| \; |X_{ss}|}{|X|} \psi(\lambda) \tag{20}$$

showing that $\psi(\lambda)$ is a divisor of $|A - \lambda E|$.

We have mentioned already above that the coefficients of the polynomial $\psi(\lambda)$ (which are identical with those of the right hand member of (11)) can be determined without computing the determinants.

4. Let us use $E_k (k = 1, 2, \ldots, n)$ as vector x and find the corresponding divisors $\psi_k(\lambda)$. It is easily seen that their common minimum multiple $\Phi(\lambda)$ either coincides with the minimum polynomial of the matrix A, or is divisible by it without rest. Indeed, assume

$$\Phi(\lambda) = g_k(\lambda)\,\psi_k(\lambda) \qquad (k = 1, \ldots, n) \tag{21}$$

Since, according to (11), $\psi_k(A)E_k = 0$, we must have

$$\Phi(A)\,E_k = 0 \qquad (k = 1, 2, \ldots, n) \tag{22}$$

or

$$\Phi(A)\,\|E_1, \ldots, E_n\| = \Phi(A)\,E = \Phi(A) = 0 \tag{23}$$

which proves the correctness of our statement.

All different roots of the equation $|A - \lambda E| = 0$ are roots of the minimum polynomial of the matrix A, therefore all different roots of the secular equation will be found among the roots of the polynomials $\psi_1(\lambda)$, $\psi_2(\lambda)$, ..., $\psi_n(\lambda)$.

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